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SOME PROBLEMS OF THE MOTION OF A PENDULUM WHEN THERE ARE HORIZONTAL VIBRATIONS OF THE POINT OF SUSPENSION†

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The motion of a mathematical pendulum whose point of suspension performs small-amplitude horizontal harmonic oscillations is considered. The non-integrability of the equation of motion of the pendulum is established. The periodic motion of the pendulum originating from a stable position of equilibrium is obtained and its stability is investigated. Unstable periodic motions originating from unstable positions of equilibrium are indicated and the separatrice surfaces asymptotic to these motions are determined. The problem of the existence and stability of periodic motions of the pendulum originating from its oscillations with arbitrary amplitude and rotations with arbitrary mean angular velocity is investigated.

A number of general problems on the existence of periodic motions of a pendulum with horizontal vibrations of the point of suspension were considered in [1]. The motion of a pendulum in the neighbourhood of resonance, when the frequency of the vibrations of the point of suspension is close to the frequency of its natural small oscillations were studied in [2, 3]. Subharmonic oscillations of a pendulum excited by horizontal oscillations of its point of suspension were investigated in [4].

1. FORMULATION OF THE PROBLEM

Suppose a pendulum has a length l and its point of suspension undergoes horizontal harmonic oscillations with amplitude a and frequency Ω. We will assume that the amplitude of the oscillations of the point of suspension of the pendulum is small compared with its length so that $\varepsilon = a/l \ll 1$. Changing to dimensionless time $\tau = \Omega t$ and frequency $\omega_0^2 = g/(\Omega^2 l)$ we can write the equation of motion of the pendulum in the form

$$q'' + \omega_0^2 \sin q = \varepsilon \sin\tau \cos q \tag{1.1}$$

where q is the angle of deflection of the pendulum from the vertical and the prime denotes differentiation with respect to τ.

Equation (1.1) can also be represented in the form of canonical equations

$$\frac{dq}{d\tau} = \frac{\partial H}{\partial p}, \quad \frac{dp}{d\tau} = -\frac{\partial H}{\partial q} \tag{1.2}$$

where we have introduced the momentum $p = q'$, while Hamilton's function has the form

$$H = H_0 + \varepsilon H_1; \quad H_0 = \frac{1}{2}p^2 - \omega_0^2 \cos q, \quad H_1 = -\sin\tau \sin q \tag{1.3}$$

When $\varepsilon = 0$, Eq. (1.1) becomes the well-known equation of the oscillations of a mathematical pendulum. Its constant solutions $q = 0 \pmod{2\pi}$ and $q = \pi \pmod{2\pi}$ correspond to the stable lower position of equilibrium and the unstable upper position of equilibrium. Those differing from constant solutions correspond either to oscillations of the pendulum with arbitrary amplitude, or to rotations with arbitrary mean angular velocity or to asymptotic motions.

Motion along separatrices separating regions of oscillations and rotations in the (q, p) plane correspond to asymptotic motions of the pendulum. We will denote by S^+ and S^- the separatrices in the upper and

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lower half-plane, respectively, and we will specify their following equations obtained by integrating (1.2) and (1.3), with $\varepsilon = 0$

$$p = \pm\frac{2\omega_0}{\operatorname{ch}\omega_0\tau}, \quad \cos q = \frac{2}{\operatorname{ch}^2\omega_0\tau} - 1 \tag{1.4}$$

where the upper sign corresponds to the S^+ curve while the lower sign corresponds to the S^- curve.

In the following sections we will investigate the motion of a pendulum for fairly small but non-zero values of ε.

2. SPLITTING OF THE SEPARATRICE AND THE NON-INTEGRABILITY OF EQ. (1.1)

If the point of suspension of the pendulum is fixed ($\varepsilon = 0$), we obtain the integral of energy $H_0 =$ const. We will show that for fairly small but non-zero values of ε, the system of equations (1.2) with Hamiltonian (1.3) has no real-analytic first integral differing from a constant. To do this we will use the results obtained in [5, 6].

Consider the function

$$J(\alpha) = \int_{-\infty}^{+\infty} (H_0, H_1)\,d\tau$$

where the Poisson bracket (H_0, H_1) is calculated on the unperturbed separatrices S^+ or S^-, while in H_1 we replace τ by $\tau + \alpha$. Using (1.3) and (1.4) we obtain

$$J(\alpha) = \pm 2\omega_0 \int_{-\infty}^{+\infty} \frac{\sin(\tau+\alpha)}{\operatorname{ch}\omega_0\tau}\left(\frac{2}{\operatorname{ch}^2\omega_0\tau} - 1\right)d\tau \tag{2.1}$$

where the upper sign corresponds to separatrice S^+ while the lower sign corresponds to S^-.

After converting the integrand in (2.1) and integrating by parts using well-known formulae [7], we obtain

$$J(\alpha) = \pm 2\pi\left(\omega_0^2 \operatorname{ch}\frac{\pi}{2\omega_0}\right)^{-1} \sin\alpha \tag{2.2}$$

It obviously follows from (2.2) that both for S^+ and S^- the function $J(\alpha)$ is not identically zero and changes sign. This indicates the splitting and intersection of both pairs of separatrices and the fact that there is no first integral of the system of equations (1.2) and (1.3) [5].

3. PERIODIC MOTIONS ORIGINATING FROM STABLE POSITIONS OF EQUILIBRIUM

Suppose the frequency ω_0 is not close to an integer. Then, by Poincaré's method [8], for fairly small values of ε there is a unique 2π-periodic solution q_* of Eq. (1.1), analytic in ε and which, when $\varepsilon = 0$, reduces to the solution $q = 0$

$$q_* = \varepsilon q_1 + \varepsilon^2 q_2 + \varepsilon^3 q_3 + \ldots \tag{3.1}$$

It follows from (1.1) and (3.1) that $q_i \equiv 0$ in (3.1) if i is even, while for odd i we have

$$q_1 = \frac{\sin\tau}{\omega_0^2 - 1}, \quad q_3 = \frac{3 - 2\omega_0^2}{24(\omega_0^2 - 1)^3}\left[\frac{3\sin\tau}{\omega_0^2 - 1} - \frac{\sin 3\tau}{\omega_0^2 - 9}\right], \ldots \tag{3.2}$$

Stability in the first approximation. We will put $q = q_* + x$. We can then write the linearized equation of perturbed motion, using (3.1) and (3.2), in the form

$$x'' + \left[\omega_0^2 + \varepsilon^2\frac{\omega_0^2 - 2}{2(\omega_0^2 - 1)^2}\sin^2\tau + O(\varepsilon^4)\right]x = 0 \tag{3.3}$$

It can be verified, by means of (1.1) and (3.1), that the terms $O(\varepsilon^4)$ in Eq. (3.3) only contain even harmonics.

When $\varepsilon = 0$ we have stability. For fairly small non-zero values of ε instability is possible for the linear differential equation (3.3) with periodic coefficients due to parametric resonance when $2\omega_0$ is equal to an even integer. But these values of ω_0 are eliminated from consideration, and hence, in the non-resonant case investigated (the frequency ω_0) is not close to an integer), the solution $q = q_*$ is stable for fairly small ε in the first approximation.

Non-linear analysis of stability. For a rigorous solution of the problem of the stability of solution (3.1), (3.2) we will use the methods for investigating Hamiltonian systems described in [10].

We will first obtain the characteristic exponents $\pm i\lambda$ for the linearized equations of perturbed motion (3.3). We will seek a solution of Eq. (3.3) in the form [11]

$$x = ze^{i\lambda\tau}$$

where the 2π-periodic function of time $z(\tau)$ and the quantity λ can be represented in the form of series in powers of ε

$$z = z_0 + \varepsilon z_1 + \varepsilon^2 z_2 + \dots, \qquad \lambda = \omega_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \dots$$

Setting up differential equations for the functions $z_i(\tau)$ and using the condition for their solutions to be periodic, we obtain an expression for the characteristic exponent of Eq. (3.3)

$$\lambda = \omega_0 + \varepsilon^2 \frac{\omega_0^2 - 2}{8\omega_0(\omega_0^2 - 1)^2} + O(\varepsilon^4) \tag{3.4}$$

In the Hamiltonian (1.3) we will make a replacement of variables, introducing perturbations of the solution $q_*(\tau), p_*(\tau) \equiv dq_*/d\tau$ by the formulae

$$q = x + q_*(\tau), \quad p = y + p_*(\tau)$$

The Hamiltonian of the perturbed motion can then be represented in the form of a series

$$H = H_2 + H_3 + H_4 + \dots \tag{3.5}$$

where H_k is a form of degree k in x, y with coefficients that are 2π-periodic in τ

$$\begin{aligned} H_2 &= \frac{1}{2}y^2 + \frac{1}{2}(\omega_0^2\cos q_* + \varepsilon\sin\tau\sin q_*)x^2 \\ H_3 &= \frac{1}{6}(-\omega_0^2\sin q_* + \varepsilon\sin\tau\cos q_*)x^3 \\ H_4 &= -\frac{1}{24}(\omega_0^2\cos q_* + \varepsilon\sin\tau\sin q_*)x^4 \end{aligned} \tag{3.6}$$

Using a linear 2π-periodic replacement of variables $x, y \to x_*, y_*$ (which differs from an identity by terms of order ε^2), the Hamiltonian H_2 can be reduced to the form $H_2 = \lambda(x_*^2 + y_*^2)/2$. If we then put

$$x_* = \sqrt{\frac{2r}{\omega_0}}\sin\varphi, \quad y_* = \sqrt{2\omega_0 r}\cos\varphi$$

the Hamiltonian of the perturbed motion can be written in the form (3.5), where

$$H_2 = \lambda r, \quad H_3 = -\frac{\sqrt{2}r\sqrt{r}\sin^3\varphi}{3\omega_0\sqrt{\omega_0}}[\varepsilon\sin\tau + O(\varepsilon^2)]$$

$$H_4 = \left(-\frac{1}{6} + O(\varepsilon^2)\right)r^2\sin^4\varphi \tag{3.7}$$

We will consider the stability of the motion (3.1), (3.2) for values of ω_0 and ε lying on curves of third and fourth order resonances, when the quantities 3λ and 4λ are respectively equal to an integer.

Calculations show that in the case of resonance $3\lambda = 1$, using a non-linear canonical transformation $\varphi, r \to \psi, \rho$, the Hamiltonian (3.5), (3.7) can be reduced to the form

$$H = \lambda\rho + \varepsilon\alpha\rho\sqrt{\rho}\cos 3\psi + O(\rho^2), \quad \alpha = \sqrt{6}/8 + O(\varepsilon) \tag{3.8}$$

Since, when $\varepsilon = 0$, the coefficient α in the resonance term in (3.8) is non-zero, on the resonance curve $3\lambda = 1$ the periodic motion (3.1), (3.2) is unstable [10], if ε is fairly small. The equation of this curve can be obtained from (3.4) and has the form

$$\omega_0 = \frac{1}{3} + \varepsilon^2 \frac{459}{512} + \ldots$$

In the case of resonances $3\lambda = k$, where k is even, the form of H_3 in the Hamiltonian (3.5), (3.7) vanishes for normalization, since there are no corresponding resonance terms in it (this follows from the structure of the form H_3 from (3.6) and the solution (3.1)). Consequently, the solution (3.1), (3.2) is Lyapunov stable on the corresponding resonance curves.

For resonances $3\lambda = k$, where k is odd and $k \geqslant 5$, the resonance terms in the form H_3 occur in terms of order ε^k; the problem of the stability of solution (3.1), (3.2) in this case has not been investigated.

Outside the curves of the third-order resonances the Hamiltonian of the perturbed motion can be reduced, by means of a non-linear canonical replacement of variables, to the following normal form:

$$H = \lambda\rho + c\rho^2 + \beta\rho^2\cos 4\psi + O(\rho^{5/2})$$

with constant coefficients β and c. The motion investigated is Lyapunov stable if $|c| > |\beta|$ and unstable if $|c| > |\beta|$ [10].

Calculations show that $c = -1/16 + O(\varepsilon^2)$. If the parameters ω_0 and ε do not lie on the fourth-order resonance curves, we have $\beta = 0$; if ω_0 and ε lie on these curves, we have $\beta = O(\varepsilon^2)$, and hence, the periodic motion (3.1), (3.2) is Lyapunov stable for fairly small values of ε both when there are fourth-order resonances and when there are no such resonances.

4. SOLUTIONS ORIGINATING FROM UNSTABLE POSITIONS OF EQUILIBRIUM

Unstable positions of equilibrium $(-\pi, 0)$ and $(\pi, 0)$ for $\varepsilon = 0$ transfer, for fairly small values of ε, by the Poincaré small-parameter method [8] into 2π-periodic solutions analytic in ε. These solutions have the form

$$q^* = \pm\pi + \frac{\varepsilon}{\omega_0^2 + 1}\sin\tau + O(\varepsilon^3), \quad p^* = \frac{\varepsilon}{\omega_0^2 + 1}\cos\tau + O(\varepsilon^3) \tag{4.1}$$

Solutions (4.1) are unstable, which follows from the continuity of the characteristic exponents with respect to ε of the corresponding linear equations of the perturbed motion.

We will define, as in [12], the real, 2π-periodic with respect to τ, analytic with respect to ξ, η, ε replacement of variables

$$q = q^* + Q(\xi, \eta, \tau, \varepsilon), \quad p = p^* + P(\xi, \eta, \tau, \varepsilon) \tag{4.2}$$

which reduces the Hamiltonian of the perturbed motion to normal form $H = H(\zeta)$, where

$$\zeta = \xi\eta, \quad H = \lambda\zeta + a_2\zeta^2 + a_3\zeta^3 + \ldots \quad (\lambda, a_2, a_3, \ldots - \text{const}) \tag{4.3}$$

The system of equations corresponding to (4.3) has the solution

$$\zeta = \text{const}, \quad \xi = \xi_0 \exp[(\partial H/\partial\zeta)\tau], \quad \eta = \eta_0 \exp[-(\partial H/\partial\zeta)\tau]$$

The equalities $\eta = 0$ and $\xi = 0$ define two-dimensional surfaces

$$q = q^* + Q(\xi, 0, \tau, \varepsilon), \quad p = p^* + P(\xi, 0, \tau, \varepsilon)$$

and

$$q = q^* + Q(0, \eta, \tau, \varepsilon), \quad p = p^* + P(0, \eta, \tau, \varepsilon)$$

consisting of solutions asymptotic to the solution q^*, p^*, respectively, as $\tau \to -\infty$ and $\tau \to +\infty$. These surfaces are called emerging and entering separatrices, respectively.

Calculations show that the normalizing transformation (4.2) is obtained as a result of carrying out the following sequence of canonical replacements of variables

$$q = q^* + x, \quad p = p^* + y \tag{4.4}$$

$$x = \frac{x' - y'}{\sqrt{2\omega_0}}, \quad y = \sqrt{\frac{\omega_0}{2}}(x' + y') \tag{4.5}$$

$$x' = x'' - \frac{\varepsilon}{\chi}(A^- x''^2 - 2A^+ x'' y'' + B^+ y''^2) \tag{4.6}$$

$$y' = y'' + \frac{\varepsilon}{\chi}(B^- x''^2 + 2A^- x'' y'' - A^+ y''^2)$$

$$\chi = 4\sqrt{2}\omega_0\sqrt{\omega_0}(\omega_0^2 + 1), \quad A^\pm = \frac{\pm\omega_0 \sin\tau + \cos\tau}{\omega_0^2 + 1}, \quad B^\pm = \frac{3\omega_0 \sin\tau \pm \cos\tau}{9\omega_0^2 + 1}$$

$$x'' = \xi - \frac{1}{48\omega_0}\xi^3 + \frac{1}{16\omega_0}\xi\eta^2 - \frac{1}{96\omega_0}\eta^3$$

$$y'' = \eta - \frac{1}{96\omega_0}\xi^3 + \frac{1}{16\omega_0}\xi^2\eta - \frac{1}{48\omega_0}\eta^3 \tag{4.7}$$

The normalized Hamiltonian then has the form

$$H = \omega_0\xi\eta + \frac{1}{16}(\xi\eta)^2 + O(\varepsilon^2) + O((\xi\eta)^{5/2}) \tag{4.8}$$

If we drop the last two terms in (4.8), the general solution of the system of equations corresponding to (4.8) is given by the equalities

$$\xi = \xi_0 e^{\kappa\tau}, \quad \eta = \eta_0 e^{-\kappa\tau}, \quad \kappa = \omega_0 + \frac{1}{8}\xi_0\eta_0 \tag{4.9}$$

The following solutions will be asymptotic to (4.9)

$$\xi = \xi_0 e^{\omega_0\tau}, \quad \eta = 0; \quad \xi = 0, \quad \eta = \eta_0 e^{-\omega_0\tau} \tag{4.10}$$

Substituting (4.10) into (4.4)–(4.7), which specify the normalizing transformation, we obtain separatrice surfaces for each of the solutions (4.1).

When $\varepsilon = 0$ the separatrices emerging from the point $(-\pi, 0)$ (emerging from $(\pi, 0)$) and entering at the point $(\pi, 0)$ (entering at $(-\pi, 0)$) coincide and represent the curve S^+ (S^-), specified by (1.4). When $\varepsilon \neq 0$ splitting of the separatrices occurs, as follows from the results obtained in Section 2.

5. PERIODIC MOTIONS ORIGINATING FROM OSCILLATIONS AND ROTATIONS AND THEIR STABILITY

We will investigate the problem of the existence and stability of periodic motions originating from oscillations of the pendulum with arbitrary amplitude and its rotations with an arbitrary mean angular velocity.

We will write the Hamiltonian H_0 from (1.3) in action-angle variables I, w, making a canonical, univalent, 2π-periodic in w replacement of variables $p, q \to I, w$ [13], which, in the region of the oscillations, has the form

$$q = 2\arcsin[k_1 \operatorname{sn}(2\pi^{-1}K(k_1)w, k_1)], \quad p = 2\omega_0 k_1(2\pi^{-1}K(k_1)w, k_1) \tag{5.1}$$

and in the region of rotations

$$q = \pm 2\operatorname{am}(\pi^{-1}K(k_2)w), \quad p = \pm 2\omega_0 k_2^{-1}\operatorname{dn}(\pi^{-1}K(k_2)w) \tag{5.2}$$

The upper and lower signs in (5.2) correspond to anticlockwise and clockwise rotations of the pendulum, respectively.

In (5.1) and (5.2) sn, cn, am and dn are elliptic sine, cosine, amplitude and delta amplitudes, and $K(k_i)$ $(i = 1, 2)$ is the complete elliptic integral of the first kind. The quantities k_i are functions of the action variable I, given by the equations

$$I = \frac{8\omega_0}{\pi}[E(k_1) - (1 - k_1^2)K(k_1)], \quad I = \frac{4\omega_0 E(k_2)}{\pi k_2}$$

where $E(k_i)$ is the complete elliptic integral of the second kind.

As a result of the replacement (5.1) or (5.2), the Hamiltonian (1.3) takes the form

$$H = H_0(I) + \varepsilon H_1(I, w, \tau), \quad H_0(I) = 2\omega_0^2\zeta \tag{5.3}$$

where $\zeta = k_1^2$ in the case of oscillations $\zeta = k_2^{-2}$ in the case of rotations, while the function $H_1(I, w, \tau)$ is obtained by substituting (5.1) or (5.2) into the expression for H_1 from (1.3).

The solution of the corresponding Hamiltonian (5.3) of the unperturbed system of equations (for $\varepsilon = 0$) can be written in the form

$$I = I_0, \quad w = \omega(I)\tau + w_0$$

where the frequency $\omega(I) = \partial H_0/\partial I$, in the case of oscillations and rotations of the pendulum, is given by the following respective expressions

$$\omega_1 = \frac{\pi\omega_0}{2K(k_1)}, \quad \omega_2 = \frac{\pi\omega_0}{k_2 K(k_2)} \tag{5.4}$$

Suppose that for a certain value of $I = I_0$, the frequency is a rational number: $\omega = rs^{-1}$. Then, in the perturbed motion we have a $2\pi s/r$-periodic solution of the form

$$I = I_0, \quad w = rs^{-1}\tau + w_0 \tag{5.5}$$

We will investigate the problems of the existence and stability of periodic solutions of the system of equations with Hamiltonian (5.3) when $\varepsilon \neq 0$, which reduces to the solution (5.5) when $\varepsilon = 0$. To do this we will use the theorem in [14], which is as follows.

Suppose $\bar{H}_1(I_0, w_0)$ is the mean value of the function H_1 on the unperturbed motion (5.5), i.e.

$$\bar{H}_1 = \frac{1}{2\pi s}\int_0^{2\pi s} H_1(I_0, rs^{-1}\tau + w_0, \tau)d\tau$$

and the following three conditions are satisfied

1. when $I = I_0$

$$\partial^2 H_0/\partial I^2 \neq 0 \tag{5.6}$$

2. a $w_0 = w^*$ exists such that

$$\partial \bar{H}_1/\partial w_0 = 0 \tag{5.7}$$

3. here

$$\partial^2\overline{H}_1/\partial w_0^2 \neq 0 \qquad (5.8)$$

Then a $2\pi s$-periodic solution of the system of equations with Hamiltonian (5.3) exists, which is analytic in ε and becomes the $2\pi s/r$-periodic solution (5.5) of the unperturbed system when the $\varepsilon = 0$. When the following inequality is satisfied (for $I = I_0$ and $w = w^*$)

$$(\partial^2\overline{H}_1/\partial w_0^2)\partial^2 H_0/\partial I^2 < 0 \qquad (5.9)$$

this periodic solution is unstable, and when the following inequalities are simultaneously satisfied

$$\frac{\partial^2\overline{H}_1}{\partial w_0^2}\frac{\partial^2 H_0}{\partial I^2} > 0, \quad 5\left(\frac{\partial^3\overline{H}_1}{\partial w_0^3}\right)^2 - 3\frac{\partial^2\overline{H}_1}{\partial w_0^2}\frac{\partial^4\overline{H}_1}{\partial w_0^4} \neq 0 \qquad (5.10)$$

it is Lyapunov stable.

The range of oscillations. From (5.4) and the expressions for the derivations of elliptic integrals from [7] we obtain

$$\frac{\partial^2 H_0}{\partial I^2} = -\frac{\pi^2[E(k_1)-(1-k_1^2)K(k_1)]}{16k_1^2(1-k_1^2)K^3(k_1)} < 0 \qquad (5.11)$$

and hence condition (5.6) of the theorem is satisfied.

We obtain an expression for the function H_1 by expanding the elliptic functions in Fourier series [7]

$$\overline{H}_1 = -\frac{\pi}{2sK^2(k_1)}\int_0^{2\pi s}\sum_{n=1}^{\infty}\frac{(2n-1)\sin[(2n-1)(rs^{-1}\tau+w_0)]\sin\tau}{\operatorname{ch}(K'(k_1)(\omega_0 s)^{-1}(2n-1)r)}d\tau \qquad (5.12)$$

where $K'(k_1) = K(1-k_1^2)$.

An analysis of the structure of the integrand in (5.12) shows that the condition $\overline{H}_1 \not\equiv 0$ is only possible when $r = 1$ and $s = 2n-1$ ($n = 1, 2, 3, \ldots$), i.e. when the frequency of the unperturbed motion is equal to $\omega_1 = (2n-1)^{-1}$. Then

$$\overline{H}_1 = -2s^{-1}\cos sw_0/\Delta_1, \quad \Delta_1 = \omega_0^2\operatorname{ch}(K'(k_1)/\omega_0)$$

Hence we have

$$\partial\overline{H}_1/\partial w_0 = 2\sin sw_0/\Delta_1$$

From condition (5.7) we obtain $2s$ different values of the variable w_0: $w_0 = w_0^* = k\pi/s$ ($k = 1, 2, \ldots, 2s$), corresponding to periodic solutions in the unperturbed motion. For these values of w_0, condition (5.8) is obviously satisfied since

$$\partial^2\overline{H}_1/\partial w_0^2\Big|_{w_0=w_0^*} = 2s(-1)^k/\Delta_1 \neq 0 \qquad (5.13)$$

Thus, by the above theorem, for sufficiently small ε there are $2s$ (s is an odd number) $2\pi s$-periodic solutions which, when $\varepsilon = 0$, reduce to the $2\pi s$-periodic solutions of the form (5.5) in which we put

$$w = \tau/s + k\pi/s \quad (k = 1,2,\ldots, 2s)$$

To investigate the problem of the stability of these oscillations we will verify conditions (5.9) and (5.10).

It follows from (5.11) and (5.13) that for even k inequality (5.9) is satisfied, which denotes that the corresponding periodic solutions are unstable. If k is odd, the first inequality of (5.10) holds, and in this case

$$\partial^3\overline{H}_1/\partial w_0^3\Big|_{w_0=w_0^*} = 0, \quad \partial^4\overline{H}_1/\partial w_0^4\Big|_{w_0=w_0^*} = 2s^3(-1)^{k+1}/\Delta_1$$

and hence, taking (5.13) into account, we have

$$5\left(\frac{\partial^3 \overline{H}_1}{\partial w_0^3}\right)^2 - 3\frac{\partial^2 \overline{H}_1}{\partial w_0^2}\frac{\partial^4 \overline{H}_1}{\partial w_0^4} > 0$$

i.e. the second condition of (5.10) is satisfied. Consequently, for odd values of k the periodic oscillations of the pendulum investigated are Lyapunov stable.

The range of rotations. In the range of rotations of the pendulum we have the inequality

$$\frac{\partial^2 H_0}{\partial I^2} = \frac{\pi^2 E(k_2)}{4(1-k_2^2)K^3(k_2)} > 0 \tag{5.14}$$

i.e. condition (5.6) of the theorem is satisfied.

The function H_1 in the unperturbed motion can be represented in the form of a Fourier series as follows:

$$H_1 = \mp\frac{2\pi^2}{k_2^2 K^2(k_2)}\sum_{n=1}^{\infty}\frac{n\sin[n(rs^{-1}\tau + w_0)]\sin\tau}{\operatorname{ch}(K'(k_2)(\omega_0 s)^{-1} nrk_2)}$$

Its average value will be non-zero only at a frequency of $\omega_2 = 1/s$ ($s = 1, 2, 3, \ldots$), and in this case

$$\overline{H}_1 = \mp\cos sw_0/(s\Delta_2), \quad \Delta_2 = \omega_0^2\operatorname{ch}(K'(k_2)k_2/\omega_0)$$

From condition (5.7) we obtain the following $2s$ values of w_0^*

$$w_0^* = k\pi/s \quad (k = 1, 2, \ldots, 2s)$$

Condition (5.8) when $w_0 = w_0^*$ is obviously satisfied since

$$\partial^2\overline{H}_1/\partial w_0^2 = \pm s(-1)^k/\Delta_2 \neq 0 \tag{5.15}$$

Hence, for sufficiently small ε there are $2s$ $2\pi s$-periodic motions of the pendulum, which become $2\pi s$-periodic rotations when $\varepsilon = 0$.

It follows from (5.14) and (5.15) that inequalities (5.9) are satisfied if the pendulum rotates in an anticlockwise direction (the upper sign) and k is odd, and also if the pendulum rotates in a clockwise direction (the lower sign) and k is even. These periodic motions of the pendulum are unstable.

If k is even (when the rotations are in an anticlockwise direction) or odd (when the rotations are in a clockwise direction), inequalities (5.10) are satisfied and, consequently, these periodic motions of the pendulum are the Lyapunov stable.

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REFERENCES

1. STRUBLE R. A. and MARLIN J. A., Periodic motion of a simple pendulum with periodic disturbance. *Q. J. Mech. Math.* **18**, 4, 405–417, 1965.
2. JEFFREYS H., The simple pendulum under periodic disturbance. *Q. J. Mech. Appl. Math.* **15**, 2, 124–128, 1959.
3. STRUBLE R. A., On the simple pendulum under periodic disturbance. *Q. J. Mech. Appl. Math.* **15**, 2, 245–251, 1962.
4. CHESHANKOV B. I., Subharmonic oscillations of a pendulum. *Prikl. Mat. Mekh.* **35**, 2, 243–248, 1971.
5. KOZLOV V. V., Integrability and non-integrability in Hamiltonian mechanics. *Usp. Mat. Nauk.* **38**, 1, 3–67, 1983.
6. ZIGLIN S. L., The splitting of separatrices, the branching of solutions and the non-existence of an integral in the dynamics of a rigid body. *Trudy Mosk. Mat. Obshchestva*, **41**, 287–303, 1980.
7. GRADSHTEIN I. S. and RYZHIK I. M., *Tables of Integrals, Sums and Products*. Nauka, Moscow, 1971.
8. MALKIN I. G., *Some Problems of the Theory of Non-linear Oscillations*. Gostekhizdat, Moscow, 1956.
9. YAKUBOVICH V. A. and STARZHINSKII V. M., *Parametric Resonance in Linear Systems*. Nauka, Moscow, 1987.
10. MARKEYEV A. P., *Libration Points in Celestial Mechanics and Cosmic Dynamics*. Nauka, Moscow, 1978.
11. ARTEM'YEV N. A., A method of determining characteristic exponents and its application to two problems of celestial mechanics. *Izv. Akad. Nauk. SSSR. Ser. Mat.* **8**, 2, 61–100, 1944.
12. MOSER J., The analytical invariants of an area-preserving mapping near a hyperbolic point. *Comm. Pure Appl. Math.* **9**, 4, 673–692, 1956.

13. MARKEYEV A. P., The stability of plane-oscillations and rotations of a satellite in a circular orbit. *Kosmich. Issled.* **13**, 3, 322–336, 1975.
14. MARKEYEV A. P. and CHURKINA N. I., Periodic Poincaré solutions of a canonical system with one degree of freedom. *Pis'ma v Astron. Zh.* **11**, 8, 634–639, 1985.

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